1 Isometric Embedding

Let M^n be an n-dimensional Riemannian manifold with metric locally given by

$$ds^2 = g_{ij}(x)dx^i dx^j,$$

where $x = (x^1, \ldots, x^n)$ are local coordinates on M.

Isometric embedding means a one-to-one C^∞-mapping

$$u : M^n \to \mathbb{R}^N$$

such that

$$< du, du >= ds^2$$

or in local coordinates

$$\sum_{\lambda=1}^{N} \frac{\partial u^\lambda}{\partial x^i}\frac{\partial u^\lambda}{\partial x^j} = g_{ij}, \quad i, j = 1, \ldots, n. \tag{1}$$

So a local isometric embedding problem is reduced to a PDE system. There are three different cases to deal with according to the number of equations and the number of unknowns. The number of equations of (1) is $\frac{n(n+1)}{2}$ and the number of unknowns is $N = n + d$. The system (1) is

(i) underdetermined if $\frac{n(n+1)}{2} < N$,

(ii) determined if $\frac{n(n+1)}{2} = N$,

(iii) overdetemined if $\frac{n(n+1)}{2} > N$.

In the determined case, there exists an analytic embedding by the following theorem.

Theorem 1.1 (Cartan-Janet,[3]). *If* $N = \frac{1}{2}n(n+1)$ *and* $g_{ij} \in C^\omega$, *then there exists a* C^ω*-solution* $u = (u^1, \ldots, u^{\frac{1}{2}n(n+1)})$.

Some of the results on the underdetermined case were obtained by J. Nash[4].

Theorem 1.2. *Any Riemannian n-manifold with C^k positive metric, where $3 \le k \le \infty$, has a C^k isometric embedding in $(\frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n)$-space, in fact in any small portion of this space.*

In overdetermined case, we consider the case of codimension one.

2 Isometric Embedding of Codimension One

Isometric embedding of codimension one is an isometric embedding

$$u : M^n \to \mathbb{R}^{n+1}. \tag{2}$$

This is determined if $n = 2$ and overdetermined if $n > 2$. The question of finding a necessary and sufficient condition for the existence of local isometric embedding (2) is reduced to the problem of solving the Gauss and Codazzi equations.

Let $(e_1, \dots, e_{n+1})$ be an adapted orthonormal frame and $\theta = (\theta^1, \dots, \theta^n)^t$ be a dual frame of $(e_1, \dots, e_n)$. For any 1-forms η and ψ, the **symmetric product** is defined by

$$\eta \circ \psi = \frac{1}{2}(\eta \otimes \psi + \psi \otimes \eta).$$

Let $I = \sum_{i=1}^{n} \theta^i \circ \theta^j = \sum_{i=1}^{n} (\theta^i)^2$ be the first fundamental form of M.

Let X be a tangent vector field on M and $Y = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x^i}$ a vector field on M which is not necessarily tangent to M. Define

$$\overline{\nabla}_X Y = \sum_{i=1}^{n+1} (X a_i) \frac{\partial}{\partial x_i}.$$

Proposition 2.1. *If X and Y are tangent vector fields to M, then $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$. So $[X, Y]$ is also a tangent vector field to M.*

Proof. If X and Y are tangent vector fields, then $X = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$. Thus

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\Big(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big) - \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\Big) \\
&= \sum_{i,j} b_j \frac{\partial}{\partial x_j}\Big(a_i\Big)\frac{\partial}{\partial x_i} + \sum_{i,j} a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
&\quad - \sum_{i,j} a_i \frac{\partial}{\partial x_i}\Big(b_j\Big)\frac{\partial}{\partial x_j} - \sum_{i,j} a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
&= \overline{\nabla}_X Y - \overline{\nabla}_Y X.
\end{aligned}$$

Since $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$ are tangent to M, so is $[X, Y]$. $\square$

Definition 2.2. For tangent vector fields X, Y and normal vector field N, the second fundamental form Π of M is defined by

$$\Pi(X, Y) = - < \overline{\nabla}_X N, Y > .$$

Proposition 2.3. *For tangent vector fields X, Y and normal vector field N, the second fundamental form has the properties :*

$$\Pi(X, Y) =< \overline{\nabla}_X Y, N >,$$

$$\Pi(X, Y) = \Pi(Y, X).$$

Proof. Since N is a normal vector field, $< Y, N >= 0$. Thus

$$\begin{aligned}
X < Y, N > &= < \overline{\nabla}_X Y, N > + < Y, \overline{\nabla}_X N > \\
&= 0.
\end{aligned}$$

By definition 2.2, we have

$$\begin{aligned}
\Pi(X, Y) &= - < \overline{\nabla}_X N, Y > \\
&= < Y, \overline{\nabla}_X N > \\
&= < \overline{\nabla}_X Y, N > .
\end{aligned}$$

Using the previous proposition, we get

$$\begin{aligned}\Pi(X,Y)-\Pi(Y,X) &= <\overline{\nabla}_X Y, N> - <\overline{\nabla}_Y X, N> \\ &= <\overline{\nabla}_X Y - \overline{\nabla}_Y X, N> \\ &= <[X,Y], N> \\ &= 0.\end{aligned}$$

□

Since Π is a symmetric 2-form on M, we write

$$\Pi = \sum_{i,j=1}^{n} h_{ij}\theta^i \otimes \theta^j,$$

where $h_{ij} = h_{ji}$. Then $h_{ij} = \Pi(e_i, e_j)$. Since (h_{ij}) is symmetric, its eigenvalues are real. Let $k_1, \ldots, k_n$ be eigenvalues. We call them the principal curvatures.

Theorem 2.4. *Let* $(\omega_i{}^j) = A^{-1}dA$*, where* $A = (e_1, \ldots, e_{n+1})$*. On* M*,*

$$\omega_i^{n+1} = \sum_{\lambda=1}^{n} h_{i\lambda}\theta^\lambda.$$

Proof. We know that $\overline{\nabla}_X e_i = \sum_{j=1}^{n+1} \omega_i^j(X)e_j$. Since ω is generated by $\theta^1, \ldots, \theta^n$, it is enough to show that $\omega_i^{n+1} = h_{i\lambda}$. Since (h_{ij}) is symmetric, $h_{i\lambda} = h_{\lambda i}$ and since $(e_1, \ldots, e_n, e_{n+1})$ is the adapted orthonormal frame, e_{n+1} is a normal vector. Therefore, we have

$$\begin{aligned}h_{i\lambda} &= \Pi(e_\lambda, e_i) \\ &= -<\overline{\nabla}_\lambda e_{n+1}, e_i> \\ &= -<\sum_{j=1}^{n+1} \omega_{n+1}^j(e_\lambda)e_j, e_i> \\ &= -\omega_{n+1}^i(e_\lambda) \\ &= \omega_i^{n+1}(e_\lambda).\end{aligned}$$

The last equality follows from the fact that ω is skew-symmetric as shown below. Since $< e_i, e_j >= \delta_{ij}$, we have

$$
\begin{aligned}
0 &= d < e_i, e_j > \\
&= < de_i, e_j > + < e_i, de_j > \\
&= < \sum_{\lambda=1}^{n+1} \omega_i^\lambda e_\lambda, e_j > + < e_i, \sum_{\lambda=1}^{n+1} \omega_j^\lambda e_\lambda > \\
&= \omega_i^j + \omega_j^i.
\end{aligned}
$$

$\square$

From now on we consider the case of $n = 3$. In order to obtain the structure equations, consider $E(4) \hookrightarrow Gl(5, \mathbb{R})$ with Maurer-Cartan form $\gamma = g^{-1}dg$ of $E(4)$. $E(4)$ is the set of all matrices $\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}$ with ${}^tAA = I$. Let $\sigma : M \longrightarrow E(4)$ be an adpated frame $\sigma(x) = (e_1, e_2, e_3, e_4)_x$. Then it follows that

$$
\begin{aligned}
\sigma^*(\gamma) &= \begin{bmatrix} 0 & 0 \\ {}^tAdX & {}^tAdA \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \theta^1 & 0 & -\omega_1^2 & -\omega_1^3 & -\eta_1 \\ \theta^2 & \omega_1^2 & 0 & -\omega_2^3 & -\eta_2 \\ \theta^3 & \omega_1^3 & \omega_2^3 & 0 & -\eta_3 \\ 0 & \eta_1 & \eta_2 & \eta_3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix},
\end{aligned}
$$

where $\eta_i = {\omega_i}^4$, $A = (e_1, \ldots, e_4)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ and

$$\omega = \begin{bmatrix} 0 & -{\omega_1}^2 & -{\omega_1}^3 \\ {\omega_1}^2 & 0 & -{\omega_2}^3 \\ {\omega_1}^3 & {\omega_2}^3 & 0 \end{bmatrix}.$$

Maurer-Cartan equation $d\gamma = -\gamma \wedge \gamma$ implies that

$$d(\sigma^*\gamma) = (-\sigma^*\gamma) \wedge (\sigma^*\gamma).$$

Thus

$$\begin{aligned} d\begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix} &= -\begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 & 0 \\ d\theta & d\omega & -{}^t(d\eta) \\ 0 & d\eta & 0 \end{bmatrix} &= -\begin{bmatrix} 0 & 0 & 0 \\ \omega \wedge \theta & \omega \wedge \omega - {}^t\eta \wedge \eta & -\omega \wedge {}^t\eta \\ \eta \wedge \theta & \eta \wedge \omega & -\eta \wedge {}^t\eta \end{bmatrix}. \end{aligned}$$

By Theorem 2.4, we have $\eta_i = \omega_i^4 = \sum_{\lambda=1}^3 h_{i\lambda}\theta^\lambda$, that is, $\eta = {}^t\theta H$.

Thus we obtain

$$\begin{aligned} d\theta &= -(\omega \wedge \theta), \\ d\omega &= -\omega \wedge \omega + {}^t\eta \wedge \eta \quad \text{(Gauss equation)}, \\ d\eta &= -\eta \wedge \omega \quad \text{(Codazzi equation)}, \\ \eta \wedge \theta &= 0, \\ \eta \wedge {}^t\eta &= 0, \\ \eta &= {}^t\theta H. \end{aligned}$$

It is enough to show that there exists the second fundamental form $\Pi = (h_{ij}) = H$ by the following theorem

Theorem 2.5 (Bonnet, [5]). *Suppose that two hypersurfaces M and $\widetilde{M} \subset \mathbb{R}^{n+1}$ have the same first and second fundamental forms. Then they are congruent.*

Let us summarize the process of solving the case of $n = 3$ as follows:

(i) Start with metric $g = I$.

(ii) Find orthonormal frame θ such that $I = \sum_{i=1}^{3} (\theta)^2$.

(iii) Find Levi-Civita connection for (ω_i^j) $i, j = 1, 2, 3$ such that $d\theta = -\omega \wedge \theta$ and ${}^t\omega = -\omega$. Then compute curvature $\Phi = d\omega + \omega \wedge \omega = {}^t\eta \wedge \eta$.

(iv) Solve the algebraic equation $\Phi = H\theta\, {}^t\theta H$ for H. Compute ${}^t\eta \wedge \eta = H\theta \wedge {}^t\theta H$. Let $\Phi = (\Phi_i^j)$. Compare both sides of $\Phi = H\theta \wedge {}^t\theta H$. Both sides are skew-symmetric. Then we obtain the following three equations.

$$\begin{aligned}(h_{22}h_{33} - h_{23}^2)\theta^2 \wedge \theta^3 &+ (h_{23}h_{13} - h_{12}h_{33})\theta^3 \wedge \theta^1 \\ &+ (h_{12}h_{23} - h_{22}h_{13})\theta^1 \wedge \theta^2 = \Phi_2^3,\end{aligned}$$

$$\begin{aligned}(h_{13}h_{23} - h_{12}h_{33})\theta^2 \wedge \theta^3 &+ (h_{11}h_{33} - h_{13}^2)\theta^3 \wedge \theta^1 \\ &+ (h_{12}h_{13} - h_{11}h_{23})\theta^1 \wedge \theta^2 = -\Phi_1^3,\end{aligned}$$

$$\begin{aligned}(h_{12}h_{23} - h_{13}h_{22})\theta^2 \wedge \theta^3 &+ (h_{13}h_{12} - h_{11}h_{23})\theta^3 \wedge \theta^1 \\ &+ (h_{11}h_{22} - h_{12}^2)\theta^1 \wedge \theta^2 = \Phi_1^2.\end{aligned}$$

In matrix form, these equations are

$$\operatorname{adj}(H) \begin{bmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{bmatrix} = \begin{bmatrix} \Phi_2^3 \\ -\Phi_1^3 \\ \Phi_1^2 \end{bmatrix}.$$

To compute $\operatorname{adj}(H) = K$, evaluate on (e_k, e_l). Let $\Phi_{ikl}^j = \Phi_i^j(e_k, e_l)$. Then

$$K = \begin{bmatrix} \Phi_{223}^3 & \Phi_{231}^3 & \Phi_{212}^3 \\ -\Phi_{123}^3 & -\Phi_{131}^3 & -\Phi_{112}^3 \\ \Phi_{123}^2 & \Phi_{131}^2 & \Phi_{112}^2 \end{bmatrix}.$$

Since $K = \mathrm{adj}(H) = (\det H)H^{-1}$,

$$\begin{aligned} H &= \frac{1}{\det H} K^{-1}, \\ \det K &= (\det H)^3 (\det H)^{-1} \\ &= (\det H)^2. \end{aligned}$$

Thus $\det H = \pm\sqrt{\det K}$. If $\det K > 0$, Gauss equation is solvable and the solution is unique up to sign and if $\det K < 0$, there is no solution.

(v) Check whether H satisfies Codazzi equation $d({}^t\theta H) = -({}^t\theta H) \wedge \omega$. If it holds, then H is a solution.

Here is a more general result of the codimension one case under some restrictions for M^n for $n \geq 5$. This result was shown by J. Vilms[6].

Let V be an n-dimensional real vector space with inner product. Let $\Lambda^2 V$ denote the $\binom{n}{2}$- dimensional space of bivectors of V. A linear map $L : V \to V$ defines a linear map $L \wedge L : \Lambda^2 V \to \Lambda^2 V$ by $(L \wedge L)(x \wedge y) = Lx \wedge Ly$. When V is taken to be the tangent space at a point of M^n, the curvature tensor R at that point can be thought of as a symmetric linear map $R : \Lambda^2 V \to \Lambda^2 V$. Letting L denote the second fundamental form operator and denoting the covariant derivative by ∇, we can express the Gauss and Codazzi equations as $R = L \wedge R$ and ∇L is symmetric. On the above setting, the problem of locally isometrically embedding into $\mathbb{R}^{n+1}$ a C^3 Riemannian manifold M^n with curvature of rank ≥ 6 is reduced to the following algebraic question: *Given a symmetric linear map $R : \Lambda^2 V \to \Lambda^2 V$, find necessary and sufficient condition in order that there exists a symmetric linear map $L : V \to V$ satisfying $R = L \wedge L$.*

Theorem 2.6 (J. Vilms[6]). *Let M^n, with $n \geq 5$, be a Riemannian manifold with nonsingular curvature tensor R. Then M^n admits local isometric imbedding into $\mathbb{R}^{n+1}$ if and only if*

(1) $R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) + R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) = 0$, *for all* $x_i \in V$, *and*

(2) $R^{ij}_{kl} R^{kp}_{iq} R^{lq}_{jp} + \frac{1}{4} R^{ij}_{kl} R^{kl}_{pq} R^{pq}_{ij} > 0$.

Moreover, if $n \equiv 3 \mod 4$*, then* (1) *can be replaced by* $\det R > 0$.

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